Teaching Calculus

Kenneth Williams

Introduction

Calculus in England is usually taught towards the end of the school career because it is considered to be a subject that uses sophisticated arguments and requires a good grounding in other topics on which it is based.

This article attempts to show that the subject can be taught earlier: that is any time after basic algebra, basic graph work and an understanding of gradients.

This can have many advantages since calculus is a subject with many applications in various fields, and many students find they need an understanding of this important subject within their later field of study or work. Just as other school topics are introduced early on and gradually extended and developed it would be ideal if calculus could be treated the same way.

The following shows how gradients and areas under curves can be introduced simply, without the usual notations and terminology: though these can be introduced as appropriate by the teacher.

Secant gradient

We begin with the simplest of curves: the well-known parabola $y = x^2$. We take two points on the curve and find the gradient of the line joining them. See Figure 1.

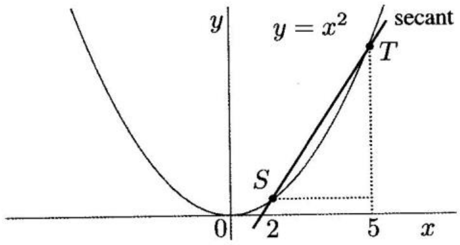

Figure 1

If the x-coordinates of T and S are 5 and 2, then the y-coordinates are 25 and 4. We then find the gradient to be

$$\frac{25-4}{5-2} = \frac{21}{3} = 7.$$

We may note that the result, 7, happens to be the sum of the x-coordinates: $5 + 2 = 7$.

This will always be so: further examples can confirm the result, and it is also easy to prove: see [1, p. 14].

So finding the gradient of any secant for $y = x^2$ could hardly be easier.

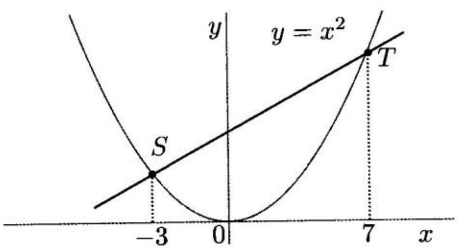

Figure 2

In Figure 2 for example, we say the gradient is $7 + -3 = 4$.

Moving point

Now suppose we allow S above to move along the curve towards T. At some point it will arrive at the origin, as shown in Figure 3.

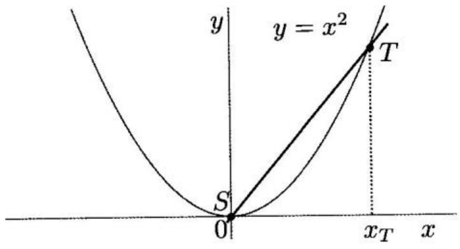

Figure 3: Origin gradient $= x$

What is the gradient of the secant now? Since the x-coordinate of S will now be zero, the secant gradient must be equal to the x-coordinate of T, x_T. We can call this the origin gradient.

The gradient of any secant through the origin is equal to the x-coordinate.

Suppose now that S moves further along the curve and merges into T (Figure 4). As this happens the secant becomes a tangent and the two x-values get closer and closer together so that the gradient will ultimately be the sum of two equal x-values. For $y = x^2$, the gradient will be $2x$.

In general, the gradient of a tangent will be twice the value of x at the point of contact, for this curve.

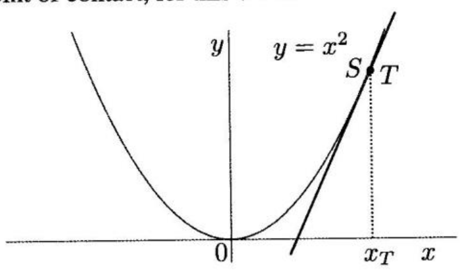

Figure 4: Tangent gradient $= 2x$

Some simple illustrations of geometrical limits can be included prior to this so that the crucial step of merging S and T will be accepted easily.

Polynomial differentiation

This method for finding tangent gradients can be used for all curves of the form $y = ax^n$, and indeed for polynomials too. For example, to find $\mathrm{d}y/\mathrm{d}x$ for $y = x^2 + 5x$ we take points $P(p, p^2 + 5p)$ and $Q(q, q^2 + 5q)$. Then the secant gradient is

$$\frac{p^2 + 5p - q^2 - 5q}{p - q} = \frac{p^2 - q^2 + 5(p - q)}{p - q} = p + q + 5.$$

Now as Q merges into P, q becomes equal to p, so the tangent gradient will be $2p + 5$, or $2x + 5$. This cancelling of $(p - q)$ will always occur because $(p - q)$ divides into $(p - q)$, $(p^2 - q^2)$, $(p^3 - q^3)$ and so on.

Ratio of gradients

Now, if the *origin gradient* equals the x-value and the *tangent gradient* equals twice the x-value then the tangent gradient is always twice the origin gradient, and this factor of 2 is the 2 in the exponent in $y = x^2$.

In fact (and it is easy to prove – see [1, p. 117]) the ratio of gradients is always equal to n in $y = ax^n$.

This ratio is useful for finding gradients of curves of this type, especially when n is fractional or negative.

For example, to find the gradient of $y^2 = 9x^3$ at the point $(4, 24)$ we note that the value of n is $3/2$, and the origin gradient is $24/4$. So we can immediately say that:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2} \times \frac{24}{4} = 9.$$

This ratio also ties up neatly with a ratio of areas which we come to next.

Area under a curve

Take a line $y = ax$. Take a point on the line and drop perpendiculars onto the axes as shown in Figure 5.

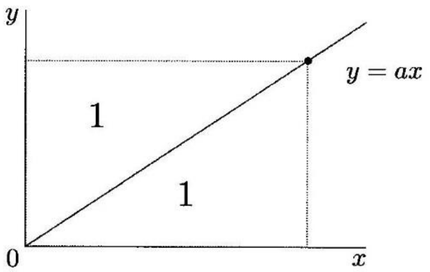

Figure 5

Then you will agree that the ratio of areas of the two triangles formed will be $1 : 1$.

Suppose we wish to draw a line from the origin such that the ratio of areas similarly formed will be $2 : 1$ rather than $1 : 1$.

How can this be done?

If the area to the left is to be twice the area to the right, it becomes clear that the answer, if it exists, is a curve with an increasing gradient.

In fact it turns out to be the curve $y = x^2$, as shown in Figure 6.

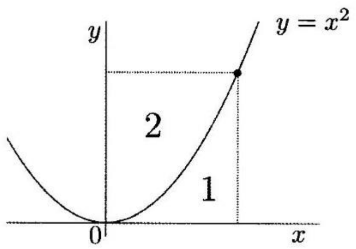

Figure 6

All the curves $y = ax^2$ will have this property.

This makes finding the area under the curve very easy because the area indicated by '1' in Figure 6 must be one third of the area of the rectangle, and the area of the rectangle is easy to find.

For example, if we require the area under $y = x^2$ from the origin up to $x = 4$ (see Figure 7), the height of the rectangle must be 4^2. Thus the rectangle area is 64 and the required area under the curve is $64/3$.

As in the case of the ratio of gradients earlier the ratio of areas here is equal to the exponent in $y = x^2$.

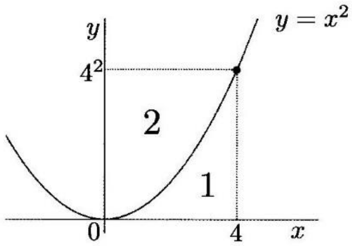

Figure 7 (not to scale)

We can of course also find areas between limits by subtraction. For example for the area under $y = x^2$ between $x = 2$ and $x = 3$ (see Figure 8), we subtract the area up to $x = 2$ from the area up to $x = 3$.

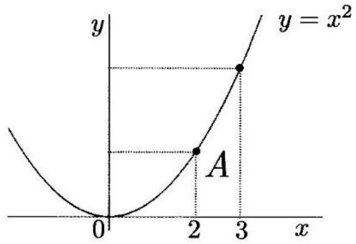

Figure 8

We get $A = 27/3 - 8/3 = 19/3$.

In fact, it is just one third of the difference of the cubes of the x-limits.

Like the gradients, this generalises to other exponents: for $y = ax^n$ the ratio of areas is $n : 1$.

So for $y = ax^3$ the ratio of areas will be $3 : 1$ and so on. A very attractive and easy to remember result.

Also, like the gradients, this applies for values of n that are fractional or negative.

Conclusion

In case the reader is not familiar with the methods and arguments usually employed to teach finding gradients of curves and areas under them it must be pointed out that this approach is very much simpler.

As mentioned at the beginning, we have used no complex notation or terminology. The method is easy to understand and the calculations are easy too.

The fundamental theorem of calculus can also be demonstrated.

The only sacrifice that has been made is the restriction to curves of the form $y = ax^n$, or polynomials. But this is no problem as an immense range of possibilities can be explored here and all the usual calculus techniques can be introduced too (differentiation of products and quotients, function of a function, etc.).

Usually students are plunged headlong into a bewildering range of techniques, notation and sophisticated arguments, but if they become familiar with calculus methods with simple functions early on the more advanced functions (exponential, logarithmic, trigonometric, etc.) will surely be easier for them.

References

1 Williams, K. (2015) *The Art of Calculus*, Inspiration Books.